

Anomalous Breaking of Anisotropic Scaling Symmetry in the Quantum Lifshitz Model

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ABSTRACT: In this note we investigate the anomalous breaking of anisotropic scaling symmetry $(t, x) \rightarrow (\lambda^z t, \lambda x)$ in a non-relativistic field theory with dynamical exponent $z = 2$. On general grounds, one can show that there exist two possible “central charges” which characterize the breaking of scale invariance. Using heat kernel methods, we compute these two central charges in the quantum Lifshitz model, a free field theory which is second order in time and fourth order in spatial derivatives. We find that one of the two central charges vanishes. Interestingly, this is also true for strongly coupled non-relativistic field theories with a geometric dual described by a metric and a massive vector field.

KEYWORDS: Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT), Anomalies in Field and String Theories

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1 Introduction

In recent years there has been considerable progress in extending the holographic correspondence to spacetimes that are not asymptotically AdS. Schroedinger [1, 2] and Lifshitz [3, 4] spacetimes were introduced as possible duals of non-relativistic strongly coupled field theories.

Holographic renormalization for these spacetimes was explored in [5] for the Schrödinger case and more recently in [6–8] for the Lifshitz case. In particular, it was suggested in [7] that the non-relativistic analog of the Weyl anomaly, derived in the context of ordinary AdS/CFT in [9], might be present for 3+1 dimensional Lifshitz spacetimes with $z = 2$. Conformal anomalies play an important role in relativistic field theories, especially in two dimensions, where various physical quantities display universal behavior which is governed by the central charge only. It is clearly of interest to explore to what extent similar results carry over to non-relativistic field theories.

Motivated by these observations, in this paper we explore the general structure of the Weyl anomaly for non-relativistic field theories in $d = 2 + 1$ dimensions with $z = 2$. The cases $d = 4$, $z = 3$ and $d = 6$ were analyzed in [10] and [11] respectively. Contrary to the relativistic case, where anomalies are present only for even dimension, we show that in the non-relativistic setting anomalies can be generated in odd dimension as well. That this is in principle possible can be quite easily seen by dimensional analysis. The analogue of the trace of the stress-tensor for non-relativistic field theories has dimension $z + d - 1$. A term with a time derivatives and b spatial derivatives, on the other hand, has dimension $az + b$. For generic z , there can only be contributions to the conformal anomaly with $a = 1$ and $b = d - 1$. Such terms can only appear in theories which explicitly break time reversal invariance, which will not be the case for the theories that we consider. For special values of z , especially for integer values of z , other values of a, b are allowed: for example, for $d = z + 1$, terms with either $(a, b) = (2, 0)$ or $(a, b) = (0, 2z)$ can appear. In particular, for the case of interest to us with $z = 2$, $d = 3$, one can have terms with either two time or four spatial derivatives. There are many such terms that one can write down, but the more detailed analysis that we give below shows that the anomaly is generated by a total of two linearly independent and non-trivial structures.

In order to show that the anomaly is indeed generated in field theory models enjoying Lifshitz symmetry, we compute the heat-kernel expansion of a fourth-order differential operator that transforms covariantly under local anisotropic rescalings. The corresponding free field theory model has various applications in condensed matter physics, see [12] and references therein. We explicitly compute the numerical coefficients that appear in front of the two possible contributions to the anomaly, and find that only one of them is nonzero.

An obvious question is then whether a similar statement holds for those purported strongly coupled field theories which are dual to Lifshitz spacetimes. We compute the two central charges using the holographic renormalization methods of [7]. Just like in the free field theory example, only one of the coefficients turns out to be non-vanishing. The results for the holographic computation perfectly agree with those of the recent paper [13], which computes the full conformal anomaly using the methods of [6] and which appeared while we were preparing this paper for submission to the ArXiv.

Remarkably, the anomaly in both models is entirely generated by terms that only contain time derivatives, even though more general structures containing space derivatives are in principle allowed. It is not clear to us whether there is a simple explanation of this fact.

The outline of this paper is as follows. In section 2, we setup and perform the field theory computation of the conformal anomaly. The corresponding analysis for Lifshitz spacetimes is presented in section 3, and in section 4 we present our conclusion. Some technical details are discussed in the appendices, such as the complete classification of the possible structures that can appear in the conformal anomaly, and an interpretation of the bulk metric and gauge field in the field theory language.

Note added: In an earlier version of this paper, we erroneously claimed that there were three instead of two possible conformal anomalies and used holographic renormalization to compute two of the three central charges. The aforementioned paper [13] correctly

pointed out that there can only be two and not three contributions to the anomaly. We corrected this mistake in this version and as a result our original anomaly computation now yields the complete anomaly.

2 Anisotropic-Scaling Anomaly in the Quantum Lifshitz Model

Scale invariance arises naturally when describing systems at a quantum critical point. Even when the underlying theory is *not* Lorentz invariant, it may still exhibit invariance under so-called Lifshitz-like scaling that treats time and space differently,

$$\vec{x} \mapsto \lambda \vec{x}, \quad t \mapsto \lambda^z t. \quad (2.1)$$

Only the specific value $z = 1$ is compatible with Lorentz symmetry. In the high-energy-physics community (2.1) is often called anisotropic scaling, though in the condensed-matter literature the word ‘anisotropy’ is typically used for *spatial* anisotropy only.

In this section we analyze the anomalous breaking of anisotropic scaling symmetry in a three-dimensional field theory known as the *quantum Lifshitz model* [12]. This model arises in the classical field theory description of three-dimensional Lifshitz points, which is why the corresponding quantized theory is called the ‘quantum Lifshitz model’. We refer the interested reader to [12] and references therein for a discussion on applications of the quantum Lifshitz model to condensed matter systems such as the so-called quantum dimer model.

We shall find that the anomalous breaking of the Lifshitz-like scaling (2.1), which we will sometimes refer to as the ‘Lifshitz anomaly’, is parametrized by two central charges. In this section we will use heat kernel methods in order to determine these two central charges in the quantum Lifshitz model.

2.1 The Quantum Lifshitz model

We consider a three dimensional free boson $\phi(t, x)$ with the following Euclidean action

$$S = \int dt d^2x N \sqrt{h} \left(\frac{1}{2} N^{-2} (\partial_t \phi)^2 + \frac{1}{2} (\Delta \phi)^2 \right), \quad (2.2)$$

where we have coupled the system to the degrees of freedom of an auxiliary three-dimensional metric with a preferred time foliation, of the form:¹

$$ds^2 = N^2 dt^2 + h_{ij} dx^i dx^j. \quad (2.3)$$

Furthermore, $\Delta = \nabla_i \nabla^i$ is constructed out of the covariant derivatives of the spatial metric h_{ij} . The metric (2.3) keeps its form under diffeomorphisms in time, $t \mapsto \tau(t)$, and in space, $x^i \mapsto \xi^i(\vec{x})$, and both leave the action (2.2) invariant if we transform N and h_{ij} as dictated by (2.1). This is one of the reasons why it is convenient to combine N and h_{ij} in an auxiliary

¹We did not include a shift N^i , because it can be removed locally by a foliation preserving diffeomorphism, e.g. $t \mapsto \tau(t)$ and $x^i \mapsto \xi^i(t, \vec{x})$.

metric, the other reason is that this structure also naturally emerges in the holographic setup.

By integrating by parts and ignoring boundary terms², the action can be written as:

$$S = \int dt d^2x N \sqrt{h} \phi D \phi, \quad (2.4)$$

where D is given by:

$$D = -\frac{1}{N\sqrt{h}} \partial_t N^{-1} \sqrt{h} \partial_t + \frac{1}{N} \Delta N \Delta \quad (2.5)$$

This model is classically invariant under local anisotropic scale transformations, that is

$$N \rightarrow e^{2\omega} N, \quad h_{ij} \rightarrow e^{2\omega} h_{ij}, \quad \phi \rightarrow \phi, \quad (2.6)$$

where ω is an arbitrary function of t and x^i . In particular, the operator D transforms as

$$D \rightarrow e^{-4\omega} D \quad (2.7)$$

The variation of the action S under a Weyl transformation is thus given by

$$\delta S = \int dt d^2x \left(\delta N \frac{\delta S}{\delta N} + \delta h_{ij} \frac{\delta S}{\delta h_{ij}} \right) = \int dt d^2x \delta \rho \left(2N \frac{\delta S}{\delta N} + 2h_{ij} \frac{\delta S}{\delta h_{ij}} \right). \quad (2.8)$$

Defining the energy density \mathcal{E} and momentum flux (spatial stress tensor) Π_{ij} as

$$\mathcal{E} = \frac{2}{N\sqrt{h}} N^2 \frac{\delta S}{\delta N^2}, \quad \Pi^{ij} = \frac{2}{N\sqrt{h}} \frac{\delta S}{\delta h_{ij}}, \quad (2.9)$$

we see that local anisotropic scale invariance implies:

$$2\mathcal{E} + \Pi_i^i = 0, \quad (2.10)$$

which is the non-relativistic analog of the tracelessness condition $T^a_a = 0$.

We will prove in this section that the classical anisotropic conformal invariance of this model is broken at the quantum level. The quantum expectation values of the energy density $\langle \mathcal{E} \rangle$ and the spatial stress tensor $\langle \Pi^{ij} \rangle$ are given by (2.9) when one replaces S by the effective action $W = -\log Z$. In the presence of an anomaly, the right-hand side of

$$2\langle \mathcal{E} \rangle + \langle \Pi_i^i \rangle = \mathcal{A} \quad (2.11)$$

is non-zero, where \mathcal{A} is the anomaly. Equivalently, one may express the anomaly directly in terms of the variation of the (renormalized) effective action W ,

$$\delta W = \int N \sqrt{h} \delta \rho \mathcal{A}, \quad (2.12)$$

where $\delta \rho = \omega$ is the infinitesimal scale factor. We will show that

$$\mathcal{A} = \frac{1}{1536\pi} \frac{1}{N^2} \left(16h^{ij} N \partial_t (N^{-1} \dot{h}_{ij}) + 5(h^{ij} \dot{h}_{ij})^2 - 10h^{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li} \right) + \frac{1}{480\pi} \frac{1}{N} \nabla_i J^i, \quad (2.13)$$

²In the following, we will assume that the theory is defined on a manifold without boundary.

where a dot indicates ∂_t and $\nabla_i J^i$ is a “trivial” total derivative, by which we mean that it can be removed by adding appropriate local counterterms. In fact we will show that by adding appropriate counterterms to the action (2.2), the anomaly can be written in the simpler form

$$\mathcal{A} = \frac{1}{128\pi} \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right). \quad (2.14)$$

This anomaly will be computed using a heat-kernel expansion.

As an aside, one of the reasons for the particular interest in this model is that the ground-state wave functional is invariant under time-independent conformal transformations in space. All equal-time correlators can be computed using the machinery of a two-dimensional field theory [12, 14]. One may thus naively expect that the anomalous breaking of anisotropic scaling symmetry (2.1) is somehow related to the two-dimensional Weyl anomaly $\langle T^i_i \rangle \propto R$, see e.g. §5.A of [15]. We find, however, that this is *not* the case. The anomaly that we find involves only derivatives with respect to the *time* coordinate, whereas the two-dimensional Ricci scalar R obviously only contains spatial derivatives.

2.2 Heat-kernel expansion

The quantum effective action W for the model (2.2) can be computed explicitly, and is given by the formal expression:

$$W \propto \frac{1}{2} \ln \det(D), \quad (2.15)$$

where $\det(D)$ is the determinant of the operator D defined in equation (2.5). As usual, this determinant is not well-defined and must be regularized. We will employ ζ -function regularization. We define the generalized zeta function as

$$\zeta(s, f, D) = \text{Tr}_{L^2}(f D^{-s}), \quad (2.16)$$

where s is an arbitrary positive number and L^2 an appropriate function space on which D^{-s} is trace-class. The regularized effective action is given by [16]:

$$W = -\frac{1}{2} \zeta'(0, 1, D) - \frac{1}{2} \ln(\mu^2) \zeta(0, 1, D), \quad (2.17)$$

where $\zeta'(0, f, D) = \partial_s \zeta(s, f, D)|_{s=0}$ and μ is the usual arbitrary renormalization scale. The zeta function $\zeta(s, f, D)$ is related via a Mellin transformation to the heat kernel:³

$$K(\epsilon, f, D) = \text{Tr}_{L^2}(f e^{-\epsilon D}), \quad (2.18)$$

where f is an arbitrary function of t and x^i , and ϵ is an arbitrary positive parameter. In principle K depends on the global behavior of the operator D (the trace can be written

³In particular, the relation between the two is

$$\zeta(s, f, D) = \Gamma(s)^{-1} \int_0^\infty d\epsilon \epsilon^{s-1} K(\epsilon, f, D), \quad K(\epsilon, f, D) = \frac{1}{2\pi i} \oint ds \epsilon^{-s} \Gamma(s) \zeta(s, f, D).$$

as a sum over the spectrum of the operator, which is determined by global properties); however there is an asymptotic series of the form:

$$K(\epsilon, f, D) \sim \sum_{k=0}^{\infty} \epsilon^{\frac{k}{2}-1} \tilde{a}_k(f, D), \quad (2.19)$$

where $\tilde{a}_k(f, D)$ can be computed *locally* from N and h_{ij} . By repeating the analysis of [16] section 7.1, it is possible to show that the variation of the renormalized effective action under an infinitesimal anisotropic local scale transformation $h \rightarrow (1 + 2\delta\rho)h$, $N \rightarrow (1 + 2\delta\rho)N$, is given by ⁴:

$$\delta W = -2\tilde{a}_2(\delta\rho, D). \quad (2.20)$$

In other words, the anomaly is given by the ϵ^0 term in the heat-kernel expansion. As explained above, this will be a local functional of N and h ; we will therefore write:

$$\tilde{a}_2(f, D) = \int dt d^2x N \sqrt{h} f a_2(N, h_{ij}), \quad (2.21)$$

where $a_2(N, h_{ij})$ is a local function that depends on N and h_{ij} . The Weyl variation of the marginal heat kernel coefficient $\tilde{a}_2(1, D)$ vanishes. To see why, consider the Weyl variation of the heat kernel coefficient at order k ,

$$\left. \frac{d}{d\gamma} \right|_{\gamma=0} \tilde{a}_k(1, e^{-4\gamma f} D) = (4 - 2k) \tilde{a}_k(f, D), \quad (2.22)$$

which vanishes when $k = 2$. We used the identity for the Weyl variation of the full heat kernel,

$$\left. \frac{d}{d\gamma} \right|_{\gamma=0} K(\epsilon, 1, e^{-4\gamma f} D) = \left. \frac{d}{d\gamma} \right|_{\gamma=0} \text{Tr} \left(e^{-\epsilon \exp(-4\gamma f) D} \right) \quad (2.23)$$

$$= 4\epsilon \text{Tr} (f D e^{-\epsilon D}) \quad (2.24)$$

$$= -4\epsilon \frac{d}{d\epsilon} \text{Tr} (f e^{-\epsilon D}) \quad (2.25)$$

$$= -4\epsilon \frac{d}{d\epsilon} K(\epsilon, f, D). \quad (2.26)$$

As a consequence, we can use this property that $\delta \tilde{a}_2(1, D) = 0$ to find that, in local language,

$$N \sqrt{h} a_2(N, h_{ij}) = e^{4\omega} N \sqrt{h} a_2(e^{2\omega} N, e^{2\omega} h_{ij}) + \text{total derivatives}. \quad (2.27)$$

In other words, the anomaly itself must be Weyl invariant (up to total derivatives), and this puts very strong restrictions on the possible terms that can appear in a_2 .

⁴The factor 2 comes from the factor 4 in $D \rightarrow e^{-4\rho} D$ under scale transformations.

2.3 Analysis of the possible terms

Just by dimensional analysis (i.e. by requiring invariance under constant rescalings) we can see that there are many terms of the right dimension that can appear in the heat kernel expansion. In dimensional analysis, ∂_t has dimension two and ∂_i has dimension one; we are interested in terms of dimension four that are covariant under reparametrizations of t and reparametrizations of x^i . These reparametrizations should not mix t and x^i since that would ruin the form of D . Examples of terms of the right dimension are⁵

$$\begin{aligned} & N^{-2} \partial_t h_{ij} G^{ijkl} \partial_t h_{kl}, \\ & R^2, \quad N^{-1} \Delta N R, \quad (N^{-1} \Delta N)^2, \\ & h^{ij} h^{kl} (N^{-1} \partial_i N) (N^{-1} \partial_k N) (N^{-1} \nabla_j \partial_l N). \\ & \dots \end{aligned} \tag{2.28}$$

where $G^{ijkl} = \frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}$ is the DeWitt metric and λ an arbitrary real number (in General Relativity $\lambda = \frac{1}{D-2}$, where D is the spacetime dimension). Terms with one time derivative and two space derivatives cannot appear because of time reversal symmetry. All these terms are scale invariant, but the anomaly should also obey the Wess-Zumino consistency condition (namely the second variation of the effective action W should be symmetric). The possible consistent structures are classified in Appendix A. The most general form of the anomaly is therefore:

$$\begin{aligned} \mathcal{A} = & C_1 \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) \\ & + C_2 \left(R - \frac{1}{N} \Delta N + h^{ij} \left(\frac{1}{N} \partial_i N \right) \left(\frac{1}{N} \partial_j N \right) \right)^2 \\ & + (\text{trivial total derivatives}). \end{aligned} \tag{2.29}$$

Here the trivial total derivatives can be cancelled by appropriate counterterms.

2.4 Computation of the anomaly

As we explained above, the spectral function can be expanded as:

$$K(\epsilon, f, D) = \sum_{k \geq 0} \epsilon^{\frac{k}{2}-1} \int dt d^2 x N \sqrt{h} f(t, x) a_k(N, h_{ij}), \tag{2.30}$$

where $a_k(N, h_{ij})$ is a local function of N and h_{ij} . To evaluate this we need a suitable basis; it is customary to use the rescaled Fourier modes so that they are orthonormal with respect to the measure that includes the $N\sqrt{h}$ factor. Nevertheless, as pointed out in [17], the cyclicity of the trace allows us to use the usual flat Fourier modes. We thus find

$$K = \int \frac{d\omega d^2 k}{(2\pi)^3} \int dt d^2 x e^{-i\omega t - ikx} f e^{-\epsilon D} e^{i\omega t + ikx}. \tag{2.31}$$

⁵A complete classification can be found in Appendix A.

We can conjugate the Fourier mode to the left to get the expression

$$K = \int \frac{d\omega d^2 k}{(2\pi)^3} \int dt d^2 x f e^{-\epsilon D_2}, \quad (2.32)$$

where D_2 is obtained from D by shifting the derivatives as follows:

$$\partial_t \rightarrow \partial_t + i\omega \quad \partial_i \rightarrow \partial_i + ik_i. \quad (2.33)$$

The most singular term in the heat kernel is the one where we keep only the terms in D_2 without derivatives, leading to

$$\frac{1}{\epsilon} \tilde{a}_0(f, D) = \int \frac{d\omega d^2 k}{(2\pi)^3} \int dt d^2 x f e^{-\epsilon(N^{-2}\omega^2 + (k^2)^2)}, \quad (2.34)$$

where $k^2 \equiv h^{ij} k_i k_j$. This expression is readily evaluated to yield the first term in the heat kernel expansion:

$$\tilde{a}_0(f, D) = \frac{1}{16\pi} \int dt d^2 x N \sqrt{h} f(t, x) \quad (2.35)$$

Computing the subleading terms in the heat kernel expansion is now straightforward though somewhat involved. We shall write

$$D_2 = D_2^0 + D_2^{\text{int}} \quad (2.36)$$

where D_2^0 is the piece we isolated above that contains ω^2 and k^4 , and D_2^{int} the remainder. We then expand the exponential of D_2^{int} . It contains a factor of ϵ , but ω counts as $\epsilon^{-1/2}$ and k as $\epsilon^{-1/4}$ in the Gaussian integral, therefore D_2^{int} has a term which scales as $\epsilon^{-1/4}$, and to get to the finite term one needs to expand D_2^{int} up to fourth order so that we get terms up to k^{12} . However, the problem becomes tractable if we consider the time-derivative and space-derivative sectors separately. This is consistent because the anomaly can only have structures involving either *two* time derivatives or *four* spatial derivatives.

The two-derivative anomaly

In order to compute the two-derivative contribution to the anomaly, and in turn C_1 , it is sufficient to consider metrics that only depend on t , and not on x^a . Thus we can drop all the terms with spatial derivatives ∂_i in D_2^{int} . Moreover, by changing the coordinate t if necessary, we can take $N = 1$. With these assumptions, we have:

$$D_2^0 = \omega^2 + (k^2)^2, \quad (2.37)$$

$$D_2^{\text{int}} = -i\omega(\partial_t + \frac{1}{\sqrt{h}}\partial_t\sqrt{h}) - \frac{1}{\sqrt{h}}\partial_t\sqrt{h}\partial_t. \quad (2.38)$$

We have to expand to second order in D_2^{int} . Since D_2^{int} and D_2^0 do not commute, we use the following formula:

$$e^{A+B} = e^A + \int_{0 \leq \alpha \leq 1} d\alpha e^{\alpha A} B e^{(1-\alpha)A} + \int_{0 \leq \alpha + \beta \leq 1} d\alpha d\beta e^{\alpha A} B e^{\beta A} B e^{(1-\alpha-\beta)A} + \mathcal{O}(B^3). \quad (2.39)$$

We find the following contribution to a_2 :⁶

$$\tilde{a}_2(f, D) = \frac{-1}{1536\pi} \int dt d^2x \sqrt{h} f \left\{ 16h^{ij} \ddot{h}_{ij} + 5(h^{ij} \dot{h}_{ij})^2 - 10h^{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li} \right\} + \dots \quad (2.40)$$

where the ellipses denote possible four-derivative contributions. To reinstate N , we simply need to change $dt \rightarrow dtN$ and $\partial_t \rightarrow N^{-1}\partial_t$. We can remove the first term by adding local counterterms as explained in Appendix A, to obtain the two-derivative contribution to the anomaly:⁷

$$\frac{1}{128\pi} \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right). \quad (2.41)$$

Using (2.20) and (2.29), we see that

$$C_1 = \frac{1}{128\pi}. \quad (2.42)$$

The four-derivative anomaly

We now determine the four derivative contribution, and in turn C_2 . As explained in Appendix A, there are 6 possible terms that can appear, 5 of which are total derivatives. These structures are distinguished by a metric of the form $h_{ij} = e^{2f(x)} \delta_{ij}$ and $N = e^{g(x)}$, which can be used to simplify considerably the computation. The four-derivative contribution to the anomaly is thus

$$\mathcal{A} = \frac{1}{480\pi} \frac{1}{N} \nabla_i \left(-5(\partial^i N) R + 3(\partial^i N) \left(\frac{1}{N} \Delta N \right) + 2(\partial^j N) \left(\frac{1}{N} \nabla_j \partial^i N \right) - 5\partial^i \Delta N \right). \quad (2.43)$$

It is interesting to note that this result is a total derivative, and as predicted by the Wess-Zumino consistency condition, it is orthogonal⁸ to the non-trivial total derivative \mathcal{J} defined in equation (A.15). As a consequence, this term can be removed by a local counterterm and we conclude that

$$C_2 = 0. \quad (2.44)$$

In Appendix C we present an alternative derivation of $C_2 = 0$.

The anomaly

In summary, the Lifshitz model (2.2) exhibits an anomaly under anisotropic local scale transformations, which after the addition of appropriate counterterms is given by:

$$\mathcal{A} = \frac{1}{128\pi} \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right). \quad (2.45)$$

It is striking that the anomaly involves only time derivatives. So far, it is unclear to us why this happens. It is also in contrast to the naive expectation that the anomaly is somehow related to the trace anomaly of a two-dimensional conformal field theory, as we mentioned at the beginning of this section.

⁶This computation is in principle quite lengthy. However, since there are only few terms that can appear, one can work this out for a diagonal h_{ij} and then reconstruct the full answer.

⁷There is a relative factor -2, see (2.20).

⁸To see what we mean by ‘orthogonal’, we refer to the appendix A.

3 Holographic Calculation

In the previous section we showed that a theory with anisotropic scaling symmetry has an anisotropic scaling symmetry anomaly parametrized by two central charges, denoted by C_1 and C_2 . We computed these central charges for a particular model defined by the action (2.2). In this section we show that these central charges can be computed holographically for the Lifshitz spacetime considered in [6, 7].

3.1 The Hamilton–Jacobi Formalism

Lifshitz spacetime is a proposed gravitational dual to a field theory at a UV fixed point with anisotropic (Lifshitz-like) scaling symmetry $t \mapsto \lambda^z t$, $\vec{x} \mapsto \lambda \vec{x}$. The configuration of $(d+1)$ -dimensional Lifshitz spacetime that we consider is a solution of the field equations derived from the Einstein–Proca action,

$$S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \right) \quad (3.1)$$

The parameters of the theory are the curvature scale ℓ and the dynamical exponent z ; they are related to the cosmological constant and the Proca mass via

$$\Lambda = -\frac{z^2 + (d-2)z + (d-1)^2}{2\ell^2}, \quad m^2 = \frac{(d-1)z}{\ell^2}. \quad (3.2)$$

We shall work in units such that $16\pi G = 1$ and $\ell = 1$ henceforth. Also, the computation that we shall perform in this section shall be for the case of $d = 3$ boundary dimensions, though we would like to keep d explicit where this may be illuminating. The solution is then given by

$$ds^2 = dr^2 - e^{2zr} dt^2 + e^{2r} d\vec{x}^2, \quad A = \sqrt{-\alpha_0} e^{zr} dt. \quad (3.3)$$

with $\alpha_0 \equiv -(z-1)/z$. A shift in the radial coordinate $r \mapsto r + \log \lambda$ generates the Lifshitz scaling $t \rightarrow \lambda^z t$, $\vec{x} \rightarrow \lambda \vec{x}$.

The goal of this section is to compute the (divergent piece of the) on-shell value of the above action. We shall do this in the Hamilton–Jacobi (HJ) formalism [18, 19]. Much of this section should be seen as an extension to [7], so we refer to that work for a more detailed discussion of the HJ formalism. For instance, the ADM-like Hamiltonian associated to (3.1) was derived in [7]. The HJ equation is a differential equation for the on-shell action, i.e. solving the HJ equation will give us the on-shell action S_{cl} . We write the HJ equation $H = 0$ as⁹

$$\{S_{\text{cl}}, S_{\text{cl}}\} - \mathcal{L} = 0 \quad (3.4)$$

⁹Again, see [7] for more details. In particular, (3.4) is the Hamiltonian constraint; the momentum constraint is automatically satisfied by choosing an Ansatz for (the local part of) S_{cl} that is covariant on the radial cut-off slice.

where the brackets are given by the following expression. Let F and G be two arbitrary phase-space functionals, then the brackets are defined as

$$\{F, G\} \equiv \frac{-1}{(\sqrt{-\gamma})^2} \left[\left(\gamma_{ac} \gamma_{bd} - \frac{1}{d-1} \gamma_{ab} \gamma_{cd} \right) \frac{\delta F}{\delta \gamma_{ab}} \frac{\delta G}{\delta \gamma_{cd}} + \frac{1}{2} \gamma_{ab} \frac{\delta F}{\delta A_a} \frac{\delta G}{\delta A_b} + \frac{1}{2m^2} D_a \frac{\delta F}{\delta A_a} D_b \frac{\delta G}{\delta A_b} \right], \quad (3.5)$$

where γ_{ab} and A_a are the induced fields pulled back onto the radial cut-off slice. The brackets (3.5) were introduced in [18]; see also [7]. It should be noted that these brackets are only introduced as a short-hand notation for the ‘kinetic’ part of the Hamiltonian constraint; they are *not* Poisson brackets (or any other type of special brackets).

We split up the on-shell action S_{cl} into a local power-law divergent piece S_{loc} and a non-local piece W that is at most logarithmically divergent; W is finite in the absence of anomalies. Using the split $S_{\text{cl}} = S_{\text{loc}} + W$, we may write the HJ equation as

$$0 = \{S_{\text{loc}}, S_{\text{loc}}\} - \mathcal{L} + 2\{S_{\text{loc}}, W\} + \{W, W\}. \quad (3.6)$$

We define $\mathcal{H}_{\text{loc}} \equiv \{S_{\text{loc}}, S_{\text{loc}}\} - \mathcal{L}$, and we require that the divergent terms cancel.¹⁰ It might happen that we get a finite remainder \mathcal{H}_{rem} that cannot be removed by local counterterms. As a consequence, the HJ equations in the large r limit, where we can ignore $\{W, W\}$, give:

$$2\{S_{\text{loc}}, W\} \approx -\mathcal{H}_{\text{rem}}. \quad (3.7)$$

The symbol “ \approx ” means an equality in the large r limit. We will presently see that this is related to the anisotropic scaling anomaly in the dual non-relativistic field theory. First, we need to specify the boundary conditions in order to be able to identify which terms are divergent.

3.2 Boundary conditions and anomaly

From the field theory side, we know that the volume form has a definite scaling weight $[\text{Vol}_d] = [dt d^{d-1}x] = z + d - 1$. In the dual gravitational picture this weight is translated to a radial scaling, such that

$$z + d - 1 = [\text{Vol}_d] = \frac{\partial_r \sqrt{\gamma}}{\sqrt{\gamma}} = \partial_r \log N + \frac{1}{2} \partial_r \log \det h \quad (3.8)$$

Here, N is the lapse function and h_{ij} is the induced metric on a spatial slice of Σ_r . We assume that the spatial metric is of the form $h_{ij} = e^{2r} \hat{h}_{ij}$, where \hat{h}_{ij} has a finite limit as $r \rightarrow \infty$. It then follows that the lapse scales as $N \sim e^{zr}$. This puts a restriction on the degrees of freedom contained in the metric. It implies in particular that we must turn off the off-diagonal mode in γ_{ti} that scales as e^{2zr} ; in terms of the linearized modes discussed in [6, 7, 20], one needs to kill the c_{1i} mode. This naturally leads us to consider only deformations with a preferred time foliation, as suggested also in [6].

¹⁰Here we refer to a term F as “divergent” if $\lim_{r \rightarrow \infty} \sqrt{\gamma} F = \infty$.

Let us redefine N and h_{ij} to be the renormalized lapse and induced metric,¹¹

$$N = \lim_{r \rightarrow \infty} e^{-zr} (-\gamma^{tt})^{-1/2}, \quad (3.9)$$

$$h_{ij} = \lim_{r \rightarrow \infty} e^{-2r} \gamma_{ij}, \quad (3.10)$$

such that the renormalized volume form is given by $N\sqrt{h} = \lim_{r \rightarrow \infty} e^{-(z+d-1)r} \sqrt{\gamma}$. With these conditions, it is straightforward to see that:

$$2\sqrt{\gamma} \{S_{\text{loc}}, W\} \approx 2z N^2 \frac{\delta W}{\delta N^2} + 2h_{ij} \frac{\delta W}{\delta h_{ij}} + z \hat{A}_t \frac{\delta W}{\delta \hat{A}_t} \approx -\sqrt{\gamma} \mathcal{H}_{\text{rem}}, \quad (3.11)$$

where $\hat{A}_t = \lim_{r \rightarrow \infty} e^{-zr} A_t$. As noted in [6, 7], for $z \geq 2$ and $d = 3$ the vector field becomes an irrelevant operator, which prevents us from defining the on-shell action non-perturbatively in the sources. In particular, the linearized analysis for $z = 2$ shows the presence of a mode that shifts the value of $\alpha = A^a A_a$ and introduces logarithmic divergences in the metric:

$$\begin{aligned} \gamma_{tt} &= -e^{4r} \left(1 + (4c_3 r + c_4) + \frac{1}{12} (4c_1 - 5c_2 + 4c_2 r) e^{-4r} \right), \\ \gamma_{ij} &= e^{2r} \delta_{ij} \left(1 + (-2c_3 r + c_5) + \frac{1}{24} (4c_1 + 5c_2 + 4c_2 r) e^{-4r} \right), \\ \alpha &= -1 + 2c_3 + 2(c_1 + c_2 r) e^{-4r}. \end{aligned} \quad (3.12)$$

It is convenient to expand the counterterms in powers of $\alpha + 1$, i.e. in fluctuations of α around its background value $\alpha_0 = -1$. However, turning on the mode c_3 means that $(\alpha - \alpha_0)^n$ is not suppressed as $r \rightarrow \infty$, and an infinite set of counterterms would be required. Furthermore, the mode c_3 introduces logarithmic divergences in the metric sector, spoiling our definition of anisotropic conformal infinity (3.9). For this reason, we will turn off this mode¹² by setting $c_3 = 0$. In particular notice the important relation:

$$A_t = e^{2r} N + \text{subleading}, \quad (3.13)$$

where the subleading terms scale as e^{-4r} . In the frame field language of [6], this corresponds to $\delta(A_A) = \delta(A_a e_A^a) = 0$. Notice that with these boundary conditions, W becomes a functional of N and h only. Therefore we have:

$$\left. \frac{\partial W}{\partial N} \right|_{h=\text{const}} = \frac{\delta W}{\delta N} + \frac{\delta W}{\delta \hat{A}_t} \frac{\partial \hat{A}_t}{\partial N}, \quad (3.14)$$

where the variations on the right are unconstrained, while the variation on the left represents the total variation of W with respect to N . Therefore (3.11) becomes:

$$2N \frac{\partial W}{\partial N} + 2h_{ij} \frac{\delta W}{\delta h_{ij}} = -\sqrt{-\gamma} \mathcal{H}_{\text{rem}}. \quad (3.15)$$

¹¹Again, we do not consider a shift N^i , as it can locally be removed by a foliation-preserving diffeomorphism.

¹²As shown in [7], it is possible to consistently impose this condition when higher order non-linear corrections are considered, and we believe there is no obstruction at the full non-linear level.

By comparing with (2.11), we find that the anomaly is given by

$$\mathcal{A} = - \lim_{r \rightarrow \infty} e^{4r} \mathcal{H}_{\text{rem}}. \quad (3.16)$$

This is the holographic anomaly in the HJ formalism. In the following we will use the aforementioned boundary conditions in order to identify possible divergent terms.

3.3 Two-derivative counterterms and anomaly

The Hamiltonian constraint at the level of two spacetime derivatives is given by

$$\mathcal{H}_{\text{loc}}^{(2)} = \{S_{\text{loc}}, S_{\text{loc}}\}^{(2)} - R + \frac{1}{4} F_{ab} F^{ab} \quad (3.17)$$

Now, let us take the most general Ansatz for the on-shell action in such a way that $\mathcal{H}_a = 0$ (i.e. covariant on Σ_r). At the non-derivative level, we thus take

$$S_{\text{loc}}^{(0)} = \int d^d x \sqrt{\gamma} \mathcal{F}_0(\alpha), \quad (3.18)$$

with $\mathcal{F}_0(\alpha)$ an arbitrary function of the Lorentz scalar $\alpha \equiv A_a A^a$. This case was analyzed in detail in [7], and does not lead to anomalies in the present case. At the level of two spacetime derivatives, we take¹³

$$\begin{aligned} S_{\text{loc}}^{(2)} = \int d^d x \sqrt{\gamma} \Big\{ & \mathcal{F}_1 R + \mathcal{F}_2 D_a A_b D^a A^b + \mathcal{F}_3 D_a A_b D^b A^a + \mathcal{F}_4 (D_a A^a)^2 + \mathcal{F}_5 \partial^a \alpha \partial_a \alpha \\ & + \mathcal{F}_6 A^b D_b A_a A^c D_c A^a + \mathcal{F}_7 A^a \partial_a \alpha D_b A^b + \mathcal{F}_8 A^a D_a A^b \partial_b \alpha + \mathcal{F}_9 (A^a \partial_a \alpha)^2 \Big\}, \end{aligned} \quad (3.19)$$

where each coefficient \mathcal{F}_i is a function of α . We do not include terms that involve second derivatives, e.g. $\sim D_a D_b A_c$, since those can be expressed in terms of first-derivative terms by means of partial integration. Equation (3.17) can be solved straightforwardly by expanding the coefficient-functions around the Lifshitz background,

$$\mathcal{F}_i(\alpha) = \sum_{n \geq 0} f_{i(n)} (\alpha - \alpha_0)^n. \quad (3.20)$$

The coefficients $f_{i(n)}$ are then found by solving the equation $\mathcal{H}_{\text{loc}}^{(2)} = 0$ recursively order by order.

For the case of $z = 2$ in $d = 3$ boundary dimensions one finds a break-down of the recursive ‘descent’ equations, i.e. $\mathcal{H}_{\text{loc}}^{(2)} \neq 0$ for any choice of coefficient functions \mathcal{F}_i . Such a break-down signals the presence of an anomalous remainder \mathcal{H}_{rem} giving rise to the holographic anomaly. As explained in the introduction to this section, we should still be

¹³In order to get a general understanding of these terms, notice that the terms proportional to \mathcal{F}_i with $i = 1, 2, 3$ are of the schematic form $\gamma \gamma D A D A$, for $i = 4, 5, 6, 7$ they look like $A A \gamma D A D A$, and for $i = 7$ is $A A A A D A D A$.

able to determine the coefficient of the *divergent* part of the on-shell action in order to have a well-defined theory on the boundary. Doing so leads to the following remainder:¹⁴

$$\mathcal{H}_{\text{rem}}^{(2)} = -D_a A_b D^b A^a + \frac{1}{2}(D_a A^a)^2, \quad (3.21)$$

which is obviously affected by the ambiguity of adding finite local counterterms to the action. Using the boundary conditions described in 3.2 and the definition of the holographic anomaly (3.16), we find the following two-derivative contribution to the anomaly:

$$\mathcal{A} = \frac{\ell}{64\pi G} \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) + \dots, \quad (3.22)$$

where the ellipses denote possible four-derivative contributions. The above anomaly is indeed of the form (2.29) with:

$$C_1 = \frac{1}{128\pi} \frac{2\ell}{G} \quad (3.23)$$

We reinstated the four-dimensional Newton's constant G and the curvature length scale ℓ . Also, \dot{h}_{ij} is generically related to the extrinsic curvature by $K_{ij} = (\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i)/2N$ (and we had set the shift N_i to zero).

3.4 Four-derivative anomaly

One may repeat the above steps at the level of four derivatives.¹⁵ The four-derivative Ansatz is

$$S_{\text{loc}}^{(4)} = \int d^d x \sqrt{\gamma} \left(\mathcal{G}_1 R^2 + \mathcal{G}_2 R_{ab} R^{ab} + \mathcal{G}_3 R_{abcd} R^{abcd} + \mathcal{G}_4 \square R \right) + \dots \quad (3.24)$$

The ellipses denote terms that involve the Proca field A_a . All the terms that appear at this level are finite for our choice of boundary conditions, which means that they can only contribute with trivial total derivatives to the anomaly. In this case, we find the remainder

$$\mathcal{H}_{\text{rem}}^{(4)} = \frac{1}{8} R^2 - \frac{1}{4} R_{ab} R^{ab} + \dots \quad (3.25)$$

where the ellipses denote once again terms that involve A_a . Writing this in terms of the two-dimensional Ricci tensor gives:

$$\mathcal{H}_{\text{rem}}^{(4)} = \frac{1}{4} {}^2R_{ij} {}^2R^{ij} - \frac{1}{8} ({}^2R)^2 + \dots, \quad (3.26)$$

where these 2R and ${}^2R_{ij}$ are the *two*-dimensional Ricci scalar and tensor and we have not written down terms that involve derivatives acting on N . We can use the off-shell identity

¹⁴ To see how this comes about, one can look at the solution $S_{\text{loc}} = \int d^d x \sqrt{-\gamma} \mathcal{L}_{\text{loc}}$ for values of the dynamical exponent close to the critical value, $z \approx 2$. One finds a simple pole at $z = 2$,

$$\mathcal{L}_{\text{loc}}^{(2)} = \frac{1}{z-2} \left(-D_a A_b D^b A^a + \frac{1}{2} (D_a A^a)^2 \right) + O((z-2)^0),$$

where the residue is simply the two-derivative remainder (3.21).

¹⁵ One does not expect to find anomalous contributions that contain three derivatives (one time and two spatial), since terms that involve an odd number of time-derivatives are not invariant under time-reversal.

that relates the Ricci tensor to the Ricci scalar, ${}^2R_{ij} = \frac{1}{2}{}^2R h_{ij}$, which is specific to two dimensions. When we plug this into (3.26), we find that the Ricci-squared terms cancel and we have (equation (2.29)):

$$C_2 = 0, \quad (3.27)$$

which agrees with the field theory computation.

Notice that while we were able to extract the coefficient C_2 , we have not performed a complete analysis of the counterterms at the four-derivative level, which would be rather involved. Nevertheless, the complete answer has been computed using the results of [6] in [13], in perfect agreement with our result $C_2 = 0$. In conclusion, the holographic anomaly is given by

$$\mathcal{A} = \frac{\ell}{64\pi G} \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right). \quad (3.28)$$

4 Discussion and conclusions

In this paper we computed the anisotropic scaling anomaly of two Lifshitz theories, one defined using a standard field theory quantization of an explicit classical action (2.2), the other defined using the holographic correspondence. A precise definition of Lifshitz holography is still lacking, and a microscopic definition of the strongly coupled field theory is not known. It is therefore a priori not very meaningful to compare the two anomalies. Nevertheless, we found that the anomalies are quite similar. In both cases there are two possible central charges of which one vanishes, and as a consequence the two anomalies are directly proportional to each other. The ratio of the two anomalies, in the conventions used in this paper, is $2\ell/G$, with ℓ the curvature radius of the Lifshitz spacetime and G the 4d Newton constant. It would be interesting to evaluate this quantity in explicit string theory embeddings of Lifshitz spacetimes to see how it scales with the various integer fluxes, as this will provide some measure of the effective number of degrees of freedom of the dual field theory.

It is quite mysterious that the conformal anomaly only involves time derivatives, it is even mysterious that there exists a conformal anomaly at all. According to [21], the dynamical critical exponent is in general renormalized, and as soon as $z = 2 + \epsilon$ a conformal anomaly can no longer be written down. So either there is some unknown mechanism that protects the value of $z = 2$, or the conformal anomaly can be removed in the full quantum theory. In the latter case, one would be in the peculiar situation that one would need to include counterterms that diverge in the classical limit. Further work will be required to clarify this issue.

It is also clearly of interest to explore other systems with anisotropic scale invariance to examine whether the conformal anomaly is still of the same form. In particular, whenever one has a Lifshitz solution in a theory with Chern-Simons type terms, time reversal symmetry is broken and it is logically possible to have contributions with an odd number of time derivatives to the conformal anomaly. It is in principle straightforward to extend the analysis in appendix A to determine whether there are non-trivial terms of this type and we leave this as an exercise.

As mentioned before, one of the main uses of the conformal anomaly is that it is a relatively simple property of a field theory which sometimes gives rise to certain universal properties. For example, in the relativistic case, in $d = 2$ the conformal anomaly completely fixes the free energy at high temperatures, and the central charges also control the logarithmic divergences in the entanglement entropy in $d = 2, 4$. Whether similar universal properties also exist for non-relativistic field theories is an interesting open problem that we hope to come back to in the future.

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A Classification of possible terms in the anomaly

In this appendix we explore to what extent it is possible to remove total derivatives from the anomaly. This is achieved by adding appropriate scale invariant counterterms to the action that are not invariant under *local* scale transformations. Clearly, we can discuss the two-derivative and the four-derivative terms separately. Let us start with the former; there are only three possible scale-invariant terms that we can construct with two time derivatives:

$$h^{ij} \frac{1}{N} \partial_t \left(\frac{1}{N} \partial_t h_{ij} \right), \quad \frac{1}{N^2} (h^{ij} \dot{h}_{ij})^2, \quad h^{ij} \dot{h}_{jk} h^{kl} \dot{h}_{li}. \quad (\text{A.1})$$

It is straightforward to see that the two combinations

$$h^{ij} \frac{1}{N} \partial_t \left(\frac{1}{N} \partial_t h_{ij} \right), \quad \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right), \quad (\text{A.2})$$

are invariant under *local* scale transformations (up to total derivatives). These two terms are related by partial integration, and we now show that it is indeed possible to “partially integrate” inside the anomaly by adding an appropriate counterterm to the action. The most general form of the anomaly at the two derivative level is:

$$\delta W = \int \delta \rho \left\{ a \frac{1}{N} h^{ij} \partial_t \left(\frac{1}{N} \dot{h}_{ij} \right) + b \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) \right\}. \quad (\text{A.3})$$

The presence of the factor $\delta \rho$ prevents us from doing partial integration directly. Let us add the following counterterm to the action:

$$W' = W + c \int N \sqrt{h} \frac{1}{N^2} (h^{ij} \dot{h}_{ij})^2. \quad (\text{A.4})$$

It is then easy to check that

$$\delta W' = \int \delta \rho \left\{ (a - 8c) \frac{1}{N} h^{ij} \partial_t \frac{1}{N} \dot{h}_{ij} + (b + 8c) \frac{1}{N^2} \left(h^{ij} h^{kl} \dot{h}_{ik} \dot{h}_{jl} - \frac{1}{2} (h^{ij} \dot{h}_{ij})^2 \right) \right\}. \quad (\text{A.5})$$

Therefore we can pick $c = a/8$ and get rid of the first term, which is tantamount to integrating by parts, or discarding total derivatives in the anomaly. For instance, in the field theory analysis, we went from (2.40) to (2.41) using this procedure. In particular, we had $a = 1/48\pi$ and $b = -5/384\pi$, such that $b + 8c = a + b = 1/128\pi$.¹⁶

Let us now consider the four derivative level. In this case we are interested in terms of the form $\nabla_i J^i$ in the anomaly. We ask ourselves to what extent it is possible to remove them by adding local counterterms G to the action. Both the total derivatives and the local counterterms must be scale invariant, therefore there is only a finite number of them. Let us choose a basis:

$$J_a^i \quad a = 1, \dots, N \quad (\text{A.6})$$

$$G_b \quad b = 1, \dots, M. \quad (\text{A.7})$$

The Weyl variation of a linear combination $\sum_b q_b G_b$ can be written, after partial integration, as:

$$\delta \sum_b q_b G_b = \frac{\omega}{N} \sum_{ab} M_{ab} q_b \nabla_i J_a^i, \quad (\text{A.8})$$

If the variation of the effective action reads:

$$\delta W = \int \sqrt{h} \omega \left(\mathcal{A} + \sum_a c_a \nabla_i J_a^i \right), \quad (\text{A.9})$$

we can get rid of the total derivatives if we can solve the system of linear equations:

$$M_{ab} q_b = c_a. \quad (\text{A.10})$$

If we are to remove all the possible total derivatives that can appear, the number of rows N of the matrix M_{ab} must be less than or equal to the number of columns M , and the rank of the matrix should be maximal. It is easy to check that there are 6 possible functionally independent scale invariant currents J^i , and we choose the following basis:

$$\begin{aligned} J_1^i &= N \partial^i R & J_2^i &= (\partial^i N) R & J_3^i &= (\partial^i N) \left(\frac{1}{N} \partial_j N \right) \left(\frac{1}{N} \partial^j N \right) \\ J_4^i &= (\partial^i N) \left(\frac{1}{N} \Delta N \right) & J_5^i &= (\partial^j N) \left(\frac{1}{N} \nabla_j \partial^i N \right) & J_6^i &= \partial^i \Delta N. \end{aligned} \quad (\text{A.11})$$

¹⁶Bear in mind that there was an extra factor -2 coming from the relation between $\tilde{a}_2(\delta\rho, D)$ and the integrated anomaly, cf. (2.20).

Analogously, there are 12 functionally independent scale invariant counterterms, and we choose the basis:

$$\begin{aligned}
G_1 &= R^2 & G_2 &= \Delta R & G_3 &= (\frac{1}{N}\Delta N)R \\
G_4 &= (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N)R & G_5 &= ((\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N))^2 & G_6 &= (\frac{1}{N}\partial_i N)(\frac{1}{N}\partial^i N)(\frac{1}{N}\Delta N) \\
G_7 &= (\frac{1}{N}\Delta N)^2 & G_8 &= (\frac{1}{N}\partial^i N)(\frac{1}{N}\partial_j N)(\frac{1}{N}\nabla_i \partial_j N) & G_9 &= (\frac{1}{N}\partial^i N)\frac{1}{N}\partial_i \Delta N \\
G_{10} &= \frac{1}{N}\nabla_i \partial_j N \frac{1}{N}\nabla^i \partial^j N & G_{11} &= \frac{1}{N}\Delta^2 N & G_{12} &= \frac{1}{N}\partial^i N \partial_i R
\end{aligned} \tag{A.12}$$

While we have many more possible counterterms than currents, it is important to stress that not all the counterterms are independent, since we can always partially integrate inside the action. This means that some linear combinations of counterterms will have the same Weyl transformation. Furthermore, there can be Weyl invariant combinations of counterterms that do not help in removing total derivatives from the anomaly.

By taking the Weyl variation of the 12 terms G_b , it is straightforward to compute the matrix M , which is given by:

$$M_{ab} = \begin{pmatrix} -4 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ -4 & -2 & -2 & -4 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 2 & -8 & -6 & 0 & -5 & -6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & -8 & 2 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 4 & 0 & -2 & 4 & -4 & 0 & 0 \\ 0 & -2 & -2 & 0 & 0 & 0 & 4 & 0 & -4 & 4 & 0 & 2 \end{pmatrix} \tag{A.13}$$

It is easily checked that M does *not* have maximal rank (which would be 6), but it has rank 5. In fact, M has a 7 dimensional space of null vectors, which is spanned by the 6 total derivatives $\nabla_i J^i$ and a Weyl invariant term:

$$\delta \int \sqrt{h} \nabla_i J_a^i = 0, \quad \delta \int N \sqrt{h} \left(R + \frac{1}{N} \Delta N - \frac{1}{N^2} \partial_i N \partial^i N \right)^2 = 0. \tag{A.14}$$

Since the rank of M is 5, the Weyl variation of the most general counterterm spans a 5 dimensional subspace of the 6 dimensional space generated by $c_a \nabla_i J_a^i$. That means that we can find an orthonormal basis (with respect to the usual Euclidean scalar product δ_{ab}) for the currents where 5 are trivial (i.e. removable by counterterms) and 1 is non-trivial. In other words, we look for 5 vectors e_a such that $e_a = M_{ab} q_b$ admits a solution. If we now take u_a to be the null vector of the transpose of M_{ab} , it is obviously orthogonal to all the e_a since $u_a e_a = e_a M_{ab} q_b = 0$. We define the non-trivial current \mathcal{J}^i to be:

$$\mathcal{J}^i = u_a J_a^i = J_1^i - J_2^i + J_4^i + J_5^i + 2J_6^i. \tag{A.15}$$

However, we will presently show that this current does *not* obey the Wess–Zumino consistency condition, therefore it cannot appear in the anomaly.

Wess–Zumino consistency condition and \mathcal{J}^i

The goal of this section is to figure out whether all possible terms that we found above satisfy the Wess–Zumino consistency conditions. To this end, we shall compute the quantities

$$\Omega_a \equiv \delta_1 \int d^2x \sqrt{h} \omega_2 \nabla_i J_a^i - \delta_2 \int d^2x \sqrt{h} \omega_1 \nabla_i J_a^i \quad (\text{A.16})$$

$$= \int d^2x \delta_2 (\sqrt{h} J_a^i) \partial_i \omega_1 - \int d^2x \delta_1 (\sqrt{h} J_a^i) \partial_i \omega_2 \quad (\text{A.17})$$

for each $a = 1, \dots, 6$. The main idea of this analysis is to find all possible linear combinations of the Ω 's such that

$$\sum_{a=1}^6 c_a \Omega_a = 0 \quad (\text{A.18})$$

If the vector space spanned by the vectors $\{\vec{c}\}$ is six dimensional, all J_a^i 's are Wess–Zumino-consistent. If, on the other hand, this vector space is five-dimensional then we must conclude that one of the J_a^i 's is inconsistent. Since we already know that five currents can be generated by varying appropriate local scale invariant terms, these are manifestly consistent. Therefore the inconsistent current, if present, must be the non-trivial current of equation (A.15).

The way we shall carry out this computation is by first computing the first term in (A.17). The second term in (A.17) is then obtained from the first one by replacing the derivatives that act on ω_1 for derivatives that act on ω_2 by means of partial integration.

We shall start with Ω_1 . The first term in (A.17) is¹⁷

$$\delta_2 (\sqrt{h} J_1^i) \partial_i \omega_1 = \sqrt{h} (-\partial^i \omega_2 N R - \partial^i \Delta \omega_2 N) \partial_i \omega_1 \quad (\text{A.19})$$

The second term is then

$$\delta_1 (\sqrt{h} J_1^i) \partial_i \omega_2 = \sqrt{h} (-\partial^i \omega_1 N R - \partial^i \Delta \omega_1 N) \partial_i \omega_2 \quad (\text{A.20})$$

$$= \sqrt{h} (-\partial^i \omega_2 N R - \nabla^i \nabla^j (\partial_j \omega_2 N)) \partial_i \omega_1 \quad (\text{A.21})$$

$$= \sqrt{h} (-\partial^i \omega_2 N R - \partial^i \Delta \omega_2 N \quad (\text{A.22})$$

$$- \Delta \omega_2 \partial^i N - \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.23})$$

so that

$$\Omega_1 = \int d^2x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.24})$$

Similarly, from J_2^i :

$$\delta_2 (\sqrt{h} J_2^i) \partial_i \omega_1 = \sqrt{h} (\partial^i \omega_2 N R - \Delta \omega_2 \partial^i N) \partial_i \omega_1 \quad (\text{A.25})$$

$$\begin{aligned} \delta_1 (\sqrt{h} J_2^i) \partial_i \omega_2 &= \sqrt{h} (\partial^i \omega_1 N R - \Delta \omega_1 \partial^i N) \partial_i \omega_2 \\ &= \sqrt{h} (\partial^i \omega_2 N R + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \end{aligned} \quad (\text{A.26})$$

$$\Omega_2 = - \int d^2x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.27})$$

¹⁷For notational clarity, notice that the variation differs by a factor of two compared to before. For instance, $h_{ij} \rightarrow e^\omega h_{ij}$ rather than $h_{ij} \rightarrow e^{2\omega} h_{ij}$.

From J_3^i :

$$\delta_2(\sqrt{h} J_3^i) \partial_i \omega_1 = \sqrt{h} (\partial^i \omega_2 \partial_j N \partial^j N + 2 \partial_j \omega_2 \partial^i N \partial^j N) \partial_i \omega_1 \quad (\text{A.28})$$

$$\begin{aligned} \delta_1(\sqrt{h} J_3^i) \partial_i \omega_2 &= \sqrt{h} (\partial^i \omega_1 \partial_j N \partial^j N + 2 \partial_j \omega_1 \partial^i N \partial^j N) \partial_i \omega_2 \\ &= \sqrt{h} (\partial^i \omega_2 \partial_j N \partial^j N + 2 \partial_j \omega_2 \partial^i N \partial^j N) \partial_i \omega_1 \end{aligned} \quad (\text{A.29})$$

$$\Omega_3 = 0 \quad (\text{A.30})$$

From J_4^i :

$$\delta_2(\sqrt{h} J_4^i) \partial_i \omega_1 = \sqrt{h} (\partial^i \omega_2 \Delta N + 2 \partial_j \omega_2 \frac{1}{N} \partial^i N \partial^j N + \Delta \omega_2 \partial^i N) \partial_i \omega_1 \quad (\text{A.31})$$

$$\begin{aligned} \delta_1(\sqrt{h} J_4^i) \partial_i \omega_2 &= \sqrt{h} (\partial^i \omega_1 \Delta N + 2 \partial_j \omega_1 \frac{1}{N} \partial^i N \partial^j N + \Delta \omega_1 \partial^i N) \partial_i \omega_2 \\ &= \sqrt{h} (\partial^i \omega_2 \Delta N + 2 \partial_j \omega_2 \frac{1}{N} \partial^i N \partial^j N - \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \end{aligned} \quad (\text{A.32})$$

$$\Omega_4 = \int d^2 x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.33})$$

From J_5^i :

$$\delta_2(\sqrt{h} J_5^i) \partial_i \omega_1 = \sqrt{h} (\partial_j \omega_2 \nabla^i \partial^j N + \partial^i \omega_2 \frac{1}{N} \partial^j N \partial_j N + \nabla_j \partial^i \omega_2 \partial^j N) \partial_i \omega_1 \quad (\text{A.34})$$

$$\begin{aligned} \delta_1(\sqrt{h} J_5^i) \partial_i \omega_2 &= \sqrt{h} (\partial_j \omega_1 \nabla^i \partial^j N + \partial^i \omega_1 \frac{1}{N} \partial^j N \partial_j N + \nabla_j \partial^i \omega_1 \partial^j N) \partial_i \omega_2 \\ &= \sqrt{h} (\partial_j \omega_2 \nabla^i \partial^j N + \partial^i \omega_2 \frac{1}{N} \partial^j N \partial_j N - \nabla_j (\partial^i \omega_1 \partial^j N)) \partial_i \omega_1 \end{aligned} \quad (\text{A.35})$$

$$\Omega_5 = \int d^2 x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.36})$$

From J_6^i :

$$\begin{aligned} \delta_2(\sqrt{h} J_6^i) \partial_i \omega_1 &= \sqrt{h} \partial^i (2 \partial_j \omega_2 \partial^j N + \Delta \omega_2 N) \partial_i \omega_1 \\ &= \sqrt{h} (\partial^i \Delta \omega_2 N + \Delta \omega_2 \partial^i N + 2 \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \end{aligned} \quad (\text{A.37})$$

$$\begin{aligned} \delta_1(\sqrt{h} J_6^i) \partial_i \omega_2 &= \sqrt{h} \partial^i (2 \partial_j \omega_1 \partial^j N + \Delta \omega_1 N) \partial_i \omega_2 \\ &= \sqrt{h} (\partial^i \Delta \omega_2 N - \Delta \omega_2 \partial^i N) \partial_i \omega_1 \end{aligned} \quad (\text{A.38})$$

$$\Omega_6 = 2 \int d^2 x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1 \quad (\text{A.39})$$

We thus find that each Ω_a is a multiple of

$$\int d^2 x \sqrt{h} (\Delta \omega_2 \partial^i N + \partial^i (\partial_j \omega_2 \partial^j N)) \partial_i \omega_1, \quad (\text{A.40})$$

which means that there is one linear combination that does *not* satisfy the Wess–Zumino consistency conditions. In other words, all but one of the six J_a^i 's can be made consistent. Since we have already found that five of the six J_a^i 's can be canceled by variations of local terms, the one that cannot be canceled (which we called \mathcal{J}^i) must be inconsistent. We can make this more precise by noticing that the consistency equation

$$c_1 - c_2 + c_4 + c_5 + 2c_6 = 0 \quad (\text{A.41})$$

describes a five-dimensional hypersurface of consistent linear combinations $c_a J_a^i$. The set of all such c_a -vectors can be defined as those that are orthogonal to the *inconsistent* vector, v_a say, such that $c_a v_a = 0$. The inconsistent vector is

$$\vec{v} = \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & 2 \end{pmatrix} \quad (\text{A.42})$$

As a consistency check on our computations, notice that this is precisely the five-dimensional hypersurface that we mentioned above, which may be defined as all vectors that are orthogonal to u_a (as defined in (A.15)). Namely, the vector u_a is the same as the inconsistent vector, i.e. $u_a = v_a$. The fact that \mathcal{J}^i does not satisfy the Wess–Zumino condition means that it cannot appear as the variation of either local or non-local terms. The fact that there are precisely five total-derivative terms in the anomaly, all of which can be canceled by variations of local terms, was also noted in [13].

Note: In an earlier version of this paper, we had claimed that there would be three instead of two possible contributions to the Lifshitz scaling anomaly, one of which was said to be proportional to $\int \sqrt{h} \nabla_i \mathcal{J}^i$. This erroneous conclusion has been corrected in the present version of this paper.

B Role of the massive vector on the field theory side

In this section we explore the conformal invariance of the field theory Lifshitz model from a different perspective. In particular, we will show that a preferred timelike vector n^μ plays a very similar role to the vector field A^μ appearing in the bulk.

Our set-up is the following three-dimensional scalar model with critical exponent $z = 2$ [12],

$$S = \int d^2x dt \mathcal{L} = \frac{1}{2} \int d^2x dt \left(\dot{\phi}^2 - (\Delta\phi)^2 \right). \quad (\text{B.1})$$

The operator Δ is the spatial Laplacian $\Delta = \delta^{ij} \partial_i \partial_j$ and the dot denotes differentiation with respect to (imaginary) time, $\dot{\phi} = \partial_t \phi$. The Noether current density $(J_a)^b$ corresponding to the infinitesimal diffeomorphism $x^a \mapsto x^a + \varepsilon^a$ is given via the usual definition¹⁸

$$\delta_\varepsilon S = \int d^2x dt (J_a)^b \partial_b \varepsilon^a, \quad (\text{B.2})$$

The current $(J_t)^a$ generates time reparametrizations and $(J_i)^a$ generates the spatial ones; their components are given by

$$(J_t)^t = -\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\Delta\phi)^2 \quad (\text{B.3})$$

$$(J_t)^i = \partial^i \dot{\phi} \Delta\phi - \dot{\phi} \partial^i \Delta\phi \quad (\text{B.4})$$

$$(J_i)^t = -\dot{\phi} \partial_i \phi \quad (\text{B.5})$$

$$(J_i)^j = \delta_i^j \mathcal{L} + \partial_i \partial^j \phi \Delta\phi - \partial_i \phi \partial^j \Delta\phi \quad (\text{B.6})$$

¹⁸We use the notation $x^t = t$, i.e. the index a runs over $a = t, 1, 2$.

where $\partial^i = \delta^{ij}\partial_j$. One thing we see here is that $(J_t)^t = -\mathcal{E}$, where \mathcal{E} is the Hamiltonian/energy density. The conservation law reads

$$\partial_b(J_a)^b = -\partial_a\phi(\ddot{\phi} + \Delta^2\phi) \approx 0 \quad (\text{B.7})$$

The symbol \approx denotes weak equality, i.e. equality up to terms that vanish on shell. The ‘gauge’ parameter that generates the Lifshitz scaling is $\varepsilon^t = 2\varepsilon t$ and $\varepsilon^i = \varepsilon x^i$ (ε is just a small real number). The condition for scale invariance is

$$2(J_t)^t + (J_i)^i = \partial_i(-2\partial^i\phi\Delta\phi) \quad (\text{B.8})$$

whose right-hand side is not zero but a total divergence. The conserved current \mathcal{S}^a associated to scale invariance of the theory is

$$\mathcal{S}^t = 2t(J_t)^t + x^i(J_i)^t \quad \mathcal{S}^i = 2t(J_t)^i + x^j(J_j)^i \quad (\text{B.9})$$

Note that we cannot interpret the J ’s as comprising an energy momentum tensor, since it would be far from being symmetric.

If we couple the Lifshitz model to N and h_{ij} , we can easily write down the condition for conformal invariance, however the relation between the bulk (with its complete metric and the extra gauge field) and the field theory model is rather obscure. Clearly, the bulk metric does not couple to the energy momentum tensor of the field theory as defined through J_t and J_i , since that tensor is not even symmetric.

So we will now make a more precise proposal about the relation between the two. We introduce a three-dimensional metric $g_{\mu\nu}$ and a unit timelike vector n_a so that $n_a n^a = -1$. Define the projector $h_a{}^b = g_a{}^b + n_a n^b$, which is orthogonal to n^a , and

$$\Delta\phi \equiv \partial_a(h^{ab}\partial_b\phi) + \frac{1}{2}h^{ab}h^{cd}\partial_a\phi\partial_b h_{cd} \quad (\text{B.10})$$

then we can couple the Lifshitz model to h_{ab} and n_a via the covariant action

$$S = \int d^2x dt \sqrt{-g} ((n^a\partial_a\phi)^2 - (\Delta\phi)^2). \quad (\text{B.11})$$

This action is conformally invariant under

$$\delta n_a = 2\omega n_a, \quad \delta h_{ab} = 2\omega h_{ab} \quad (\text{B.12})$$

This is why it is useful to introduce $h_a{}^b$, since the three-dimensional metric itself would transform as $\delta g_{ab} = -4\omega n_a n_b + 2\omega h_{ab}$ (using the completeness relation $g_{ab} = -n_a n_b + h_{ab}$). Of course, all of this is not very profound. We have merely replaced the spatial metric h_{ij} by the projection of the metric in the plane perpendicular to unit normal n_a .

The claim is that (B.11) describes the coupling of the Lifshitz model to a metric and a gauge field in exactly the same way as one would expect from the bulk description.

With this fully covariant action, we can define a symmetric ‘‘stress tensor’’ by varying it with respect to g_{ab} . Due to the presence of n_a , this stress-tensor is not conserved though. The precise equation that expresses general covariance of the theory reads

$$2D_b \frac{\delta S}{\delta g_{ab}} = (D^a n_b) \frac{\delta S}{\delta n_b} - D_b \left(n^a \frac{\delta S}{\delta n_b} \right), \quad (\text{B.13})$$

where on the left hand side we recognize the covariant derivative of the stress-tensor. The background field n_a is the quantity that breaks the general covariance of the theory, which explains why this equation has a right-hand side.

In view of (B.12), to write the conformal anomaly in covariant variables, we also need a variation in terms of n_a

$$\sqrt{-g} \mathcal{A} = (-4n_a n_b + 2h_{ab}) \frac{\delta S}{\delta g_{ab}} + 2n_a \frac{\delta S}{\delta n_a}. \quad (\text{B.14})$$

This is exactly the same as the bulk equation with n_a playing the role of the asymptotic value of A_a . When we choose $n_t = N$, $g_{tt} = -N^2$, $g_{ti} = 0$ and $g_{ij} = h_{ij}$, the conformal anomaly becomes the expression we have been using all along.

C Alternative computation of C_2

In this section we provide an alternative computation of C_2 . Since the structure multiplying C_2 contains R , we can take $N = 1$ and assume that h_{ij} does not depend on t but does depend on x^i . With these assumptions, the ω integral separates out, yielding a factor of $\sqrt{\pi/\epsilon}$. What is left is to study the operator $\exp(-\epsilon \nabla^2)$. Now we can roughly think of the standard heat kernel expansion as the Laplace transform of the spectral density. So by taking the inverse Laplace transform we can reconstruct the spectral density. The inverse transform of ϵ^a is $s^{-1-a}/\Gamma(-a)$. Next, we can integrate this against $\exp(-\epsilon s^2)$ to obtain

$$\frac{\epsilon^{a/2} \Gamma(-a/2)}{2\Gamma(-a)}. \quad (\text{C.1})$$

This suggests that if the operator ∇ has heat kernel expansion

$$\sum_{n \geq -1} \epsilon^n L_n \quad (\text{C.2})$$

then ∇^2 has the expansion

$$\sum_{n \geq -1} \epsilon^{n/2} \frac{\Gamma(-n/2)}{2\Gamma(-n)} L_n. \quad (\text{C.3})$$

The term with $n = 1$, which would contribute to the anomaly, vanishes due to the Gamma function. Therefore the coefficient of the R^2 term vanishes¹⁹. We conclude that the coefficient C_2 in (2.29) vanishes as well.

Notice that while this method is simpler than a direct computation, it is not powerful enough to determine the total derivative terms.

D Divergent terms in the heat-kernel expansion

The methods of section 2 allow us to compute the divergent part of the heat-kernel expansion for the model considered in equation (2.2). We present the result here for completeness:

$$K(\epsilon, f, D) \sim \frac{1}{\epsilon} \tilde{a}_0(f, D) + \frac{1}{\sqrt{\epsilon}} \tilde{a}_1(f, D) + O(\epsilon^0), \quad (\text{D.1})$$

¹⁹See also [22] for a rigorous proof of this statement.

where

$$\tilde{a}_0(f, D) = \frac{1}{16\pi} \int dt d^2x N \sqrt{h} f(t, x), \quad (\text{D.2})$$

$$\tilde{a}_1(f, D) = \frac{1}{48\pi^{3/2}} \int dt d^2x N \sqrt{h} f(t, x) \left(R - \frac{1}{N} \Delta N \right). \quad (\text{D.3})$$

References

- [1] D. T. Son, “Toward an ads/cold atoms correspondence: A geometric realization of the schrodinger symmetry”, *Phys.Rev.* **D78** (2008) 046003, [0804.3972](#).
- [2] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic cfts”, *Phys.Rev.Lett.* **101** (2008) 061601, [0804.4053](#).
- [3] S. Kachru, X. Liu, and M. Mulligan, “Gravity duals of lifshitz-like fixed points”, *Phys. Rev. D* **78** (2008) 106005, [0808.1725](#).
- [4] P. Koroteev and M. Libanov, “On existence of self-tuning solutions in static braneworlds without singularities”, *JHEP* **0802** (2008) 104, [0712.1136](#).
- [5] M. Guica, K. Skenderis, M. Taylor, and B. C. van Rees, “Holography for schrodinger backgrounds”, *JHEP* **1102** (2011) 056, [1008.1991](#).
- [6] S. F. Ross, “Holography for asymptotically locally lifshitz spacetimes”, *Class.Quant.Grav.* **28** (2011) 215019, [1107.4451](#), * Temporary entry *.
- [7] M. Baggio, J. de Boer, and K. Holsheimer, “Hamilton-jacobi renormalization for lifshitz spacetime”, [1107.5562](#).
- [8] R. B. Mann and R. McNees, “Holographic renormalization for asymptotically lifshitz spacetimes”, *JHEP* **1110** (2011) 129, [1107.5792](#).
- [9] M. Henningson and K. Skenderis, “The holographic weyl anomaly”, *JHEP* **9807** (1998) 023, [hep-th/9806087](#).
- [10] I. Adam, I. V. Melnikov, and S. Theisen, “A non-relativistic weyl anomaly”, *JHEP* **0909** (2009) 130, [0907.2156](#).
- [11] P. R. S. Gomes and M. Gomes, “On ward identities in lifshitz-like field theories”, [1112.3887](#).
- [12] E. Ardonne, P. Fendley, and E. Fradkin, “Topological order and conformal quantum critical points”, *Annals Phys.* **310** (2004) 493–551, [cond-mat/0311466](#).
- [13] T. Griffin, P. Horava, and C. M. Melby-Thompson, “Conformal lifshitz gravity from holography”, [1112.5660](#).
- [14] G. Grinstein, “Anisotropic sine-gordon model and infinite-order phase transitions in three dimensions”, *Phys. Rev. B* **23** May (1981) 4615–4630.
- [15] P. Di Francesco, P. Mathieu, and D. Senechal, “Conformal field theory”, 1997.
- [16] D. V. Vassilevich, “Heat kernel expansion: User’s manual”, *Phys.Rept.* **388** (2003) 279–360, [hep-th/0306138](#).
- [17] A. Ceresole, P. Pizzochero, and P. van Nieuwenhuizen, “The curved space trace, chiral and einstein anomalies from path integrals, using flat space plane waves”, *Phys.Rev.* **D39** (1989) 1567.

- [18] J. de Boer, E. P. Verlinde, and H. L. Verlinde, “On the holographic renormalization group”, *JHEP* **0008** (2000) 003, [hep-th/9912012](#).
- [19] D. Martelli and W. Mueck, “Holographic renormalization and ward identities with the hamilton-jacobi method”, *Nucl.Phys.* **B654** (2003) 248–276, [hep-th/0205061](#).
- [20] S. F. Ross and O. Saremi, “Holographic stress tensor for non-relativistic theories”, *JHEP* **0909** (2009) 009, [0907.1846](#).
- [21] A. Adams, A. Maloney, A. Sinha, and S. E. Vazquez, “ $1/n$ effects in non-relativistic gauge-gravity duality”, *JHEP* **0903** (2009) 097, [0812.0166](#).
- [22] P. B. Gilkey, “The spectral geometry of the higher order laplacian”, *Duke Mathematical Journal* **47** (1980), no. 3, 511–528.